Quantum Teleportation of Dynamics and Effective Interactions Between Remote Systems

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Most protocols for Quantum Information Processing consist of a series of quantum gates, which are applied sequentially. In contrast, interactions, for example between matter and fields, as well as measurements such as homodyne detection of light, are typically continuous in time. We show how the ability to perform quantum operations continuously and deterministically can be leveraged for inducing non-local dynamics between two separate parties. We introduce a scheme for the engineering of an interaction between two remote systems and present a protocol which induces a dynamics in one of the parties, which is controlled by the other one. Both schemes apply to continuous variable systems, run continuously in time and are based on real-time feedback.

Most protocols in Quantum Information Science (QIS) are discrete in the sense that they consist of a sequence of unitary operations and measurements. Schemes for quantum teleportation or dense coding are typical examples. These elementary protocols are the building blocks of other applications such as quantum repeaters or quantum computing. However, some implementations are intrinsically continuous. The most prominent example are atomic ensembles interacting with light, where schemes based on the continuous detection of quadrature operators are realized [1–3]. In this system, protocols can be performed that are intrinsically deterministic and continuous in time. Here we address the question how this property can be exploited by designing primitives that take advantage of this fact. Continuous schemes have been devised in several subfields of QIS, for example for phase estimation [4, 5], error correction [6] and for the preparation and protection of quantum states [7–10] in particular in the context of dissipative schemes [11–17]. In this letter, we introduce two protocols which achieve a qualitatively new goal - to control and transmit quantum evolutions between remote locations. We present two related protocols, where we consider two remote systems which evolve according to a given local dynamics. The systems can not interact directly, but they can exchange quantum states and classical information. One scheme implements an effective non-local dynamics, where the two remote systems evolve as if they were interacting with each other. The other protocol realizes the quantum teleportation of a time evolution, which uses the dynamics of one system to steer the evolution of the other.

The setup under consideration is shown in Fig. 1a. Two spin ensembles interact with a propagating light field, which is constantly measured. By performing real-time feedback on both samples, one can establish an effective interaction between the two separate systems. Since this is done continuously, the dynamics of the two systems corresponds to the evolution under the desired interaction Hamiltonian at any instant of time. Remark-
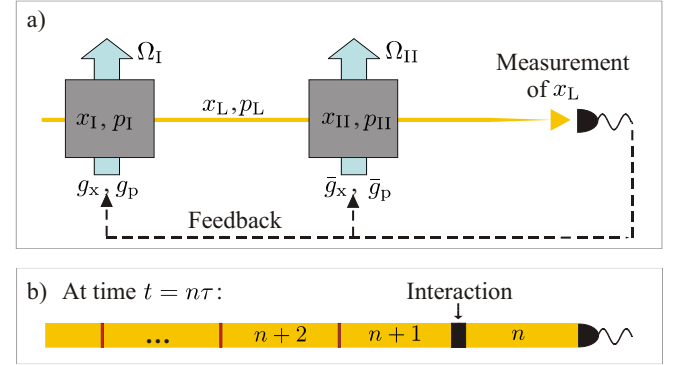

FIG. 1: Dynamical teleportation and creation of an effective interaction between two remote systems. a) The Setup consists of two atomic ensembles in constant magnetic fields oriented along $\hat{x}$. Light propagating along $\hat{z}$ interacts with both samples and is continuously measured. The result is fed back instantaneously to the atoms. b) Illustration of the interaction between the atomic system and the light field in terms of discretized spatially localized light modes.

ably, this scheme results ideally in a joint unitary evolution of the two remote atomic systems. This is the case even though the protocol is based on measurements yielding random outcomes and therefore random projections of the states involved (see also [18]). We show that using a quantum nondemolition (QND) interaction between spins and light, any Hamiltonian, which is quadratic in the atomic operators can be realized by tuning the feedback operation only, i.e. the scheme does not require the variation of the system parameters. In the ideal case, any quadratic Hamiltonian can be implemented perfectly.

The second protocol realizes a continuous quantum teleportation. Quantum teleportation [19] offers a practical solution to the delicate task of transmitting quantum states [S1]. It is a prerequisite for quantum networks [21, 22] and a primitive for quantum computa-

tion [23]. A standard teleportation scheme consists of three separate steps, which involve (i) establishing a highly entangled link between the sender and the receiver, (ii) a projective measurement, which destroys the state to be teleported, and (iii) the recovery of the state on the receiver's side by applying a feedback operation. Here, we consider a continuous process involving only weakly entangled states and measurements which disturb the quantum state only little at each instant of time. In contrast to previous approaches, which transmit a (static) quantum state by means of a single feedback operation at the end of the protocol, we consider the transmission of a whole time evolution using real-time feedback. To illustrate this point, we consider the continuous teleportation between two parties, Bob and Charlie, when a time dependent magnetic field is applied to Charlie's system. The resulting displacements on Charlie's side translate into a corresponding evolution on Bob's side, whose system evolves as if it was placed in a time dependent magnetic field. More specifically, it evolves as if interacting with a field whose time dependence is determined by Charlie's evolution.

We consider two separate systems that are characterized in terms of continuous variables x_I, p_I and x_{II}, p_{II}, which commute canonically $[x, p] = i$. We assume that both systems can be rotated locally and interact with a propagating auxiliary bosonic system via a QND interaction (Eq. (2)). For concreteness, we consider two atomic spin ensembles interacting with coherent light [24]. The free evolution of the atomic system is governed by

$$H_A = \frac{\Omega_I}{2}\left(x_I^2 + p_I^2\right) + \frac{\Omega_{II}}{2}\left(x_{II}^2 + p_{II}^2\right),$$

which describes the atomic rotation (Larmor precession) in homogeneous magnetic fields along $\hat{x}$ with Larmor frequencies Ω_I and Ω_{II} (see Fig. 1a). In the following, we use transformed atomic variables

$$\begin{pmatrix}\tilde{x}_{I/II} \\ \tilde{p}_{I/II}\end{pmatrix} = R(\Omega_{I/II})\begin{pmatrix}x_{I/II} \\ p_{I/II}\end{pmatrix},$$

$$R(\Omega_{I/II}) = \begin{pmatrix}\cos(\Omega_{I/II}t) & -\sin(\Omega_{I/II}t) \\ \sin(\Omega_{I/II}t) & \cos(\Omega_{I/II}t)\end{pmatrix}\begin{pmatrix}x_{I/II} \\ p_{I/II}\end{pmatrix}, (1)$$

which rotate at the Larmor frequencies of the respective fields. The light propagates along $\hat{z}$, passing both ensembles. We adopt here a discretized one-dimensional model. The interaction time T is divided into N infinitesimally small time steps of length τ and the incoming light is accordingly described in terms of N short pulse pieces. The quadratures associated with the nth localized light mode are denoted by $x_{L,n}$ and $p_{L,n}$ with $[x_{L,n}, p_{L,n'}] = i\delta_{n,n'}$. The nth infinitesimal pulse piece interacts with the atoms during the time window from $t = (n-1)\tau$ to $t = n\tau$ (see Fig. 1b) according to the interaction Hamiltonian [25]

$$H_{QND,n} = \frac{\kappa}{\sqrt{N\tau}}\left(p_I(n\tau) + p_{II}(n\tau)\right)p_{L,n}, \quad (2)$$

where κ is a dimensionless coupling constant. Such a QND interaction can for example be realized using a

Faraday interaction in atomic vapors [26, 27] or in optomechanical systems [28]. The total Hamiltonian is given by $H = H_L + H_A + \sum_{n=1}^{N} H_{QND,n}$, where H_L accounts for the free propagation of the light. By virtue of the consecutive QND-interactions shown in Fig. 1a, the quantum state of the atoms is mapped to the x-quadrature of the light field. The corresponding discrete input-output relation for an infinitesimal time step is given by

$$x_{L,n}^{\text{out}} = x_{L,n}^{\text{in}} + \frac{\kappa}{\sqrt{N}}\vec{V}_L^T(n\tau)\vec{R}_A([n-1]\tau), \quad (3)$$

with

$$\vec{V}_L^T(t) = \left(-\sin(\Omega_I t), \cos(\Omega_I t), -\sin(\Omega_{II} t), \cos(\Omega_{II} t)\right),$$
$$\vec{R}_A^T(t) = \left(\tilde{x}_I(t), \tilde{p}_I(t), \tilde{x}_{II}(t), \tilde{p}_{II}(t)\right),$$

where $\Omega\tau \ll 1$ has been assumed [29]. The p-quadrature is conserved $p_{L,n}^{\text{out}} = p_{L,n}^{\text{in}}$. The atomic evolution during an infinitesimal interval τ is given by

$$\begin{pmatrix}\tilde{x}_{I/II}(n\tau) \\ \tilde{p}_{I/II}(n\tau)\end{pmatrix} = \begin{pmatrix}\tilde{x}_{I/II}([n-1]\tau) \\ \tilde{p}_{I/II}([n-1]\tau)\end{pmatrix} + \frac{\kappa}{\sqrt{N}}\begin{pmatrix}\cos(\Omega_{I/II}n\tau) \\ \sin(\Omega_{I/II}n\tau)\end{pmatrix}p_{L,n}^{\text{in}}.$$

We assume that x_L is continuously measured and that the result is instantaneously fed back onto the atoms by performing a conditional displacement operation. We apply here feedback operations with temporally modulated gain factors $\frac{1}{\sqrt{N}}g_{x,I/II}(t)$ and $\frac{1}{\sqrt{N}}g_{p,I/II}(t)$ such that

$$\begin{pmatrix}\tilde{x}_{I/II}^{\text{fin}}(n\tau) \\ \tilde{p}_{I/II}^{\text{fin}}(n\tau)\end{pmatrix} = \begin{pmatrix}\tilde{x}_{I/II}(n\tau) \\ \tilde{p}_{I/II}(n\tau)\end{pmatrix} + \frac{1}{\sqrt{N}}\begin{pmatrix}g_{x,I/II}(n\tau) \\ g_{p,I/II}(n\tau)\end{pmatrix}x_{L,n}^{\text{out}}.(4)$$

We show now how this setup can be used to establish an arbitrary quadratic interaction between two ensembles. Using suitable local rotations [30], any interaction Hamiltonian for two continuous variable systems, which is quadratic in the system operators, can be expressed as

$$H = \gamma\left(\mu H_A + \nu H_P\right) = \gamma\left(Z x_1 p_2 + \frac{1}{Z}p_1 x_2\right), \quad (5)$$

where $\mu = \frac{1}{2}\left(Z + \frac{1}{Z}\right)$ and $\nu = \frac{1}{2}\left(Z - \frac{1}{Z}\right)$ [31]. γ characterizes the overall coupling strength of the interaction while Z parametrizes the imbalance between the active (entanglement creating) and the passive (energy conserving) components, H_A and H_P [32]. After the light-matter interactions, the measured quadrature x_L contains information on both ensembles (see Eq. (3) above). Feedback of x_L according to Eq. (4), leads therefore to terms which correspond to the evolution under both, local and interaction Hamiltonians. The feedback of information of each ensemble onto itself leads to an effective evolution according to local squeezing Hamiltonians. In order to suppress these contributions, we use the fact that the quadratures of each ensemble are mapped to the light field with an oscillatory time dependence. For implementing a purely non-local evolution, the gain function for ensemble I (II)

is chosen to oscillate with Ω_{II} (Ω_I), such that information is transferred with high efficiency between the ensembles, while contributions due to the feedback from the samples to themselves are out of phase and average out. This becomes apparent by considering the time varying gain functions $g_{x,I/II}(t) = \frac{1}{\sqrt{N}} g_{a/b} \sin(\Omega_{a/b} t)$ and $g_{p,I/II}(t) = \frac{1}{\sqrt{N}} g_{b/a} \cos(\Omega_{a/b} t)$. In the continuous limit,

$$
\begin{pmatrix} \dot{\tilde{x}}_I^{\text{fin}}(t) \\ \dot{\tilde{p}}_I^{\text{fin}}(t) \end{pmatrix} = \frac{\kappa}{T} \left[M_{g_a,g_I}^{\Omega_a,\Omega_I}(t) \begin{pmatrix} \tilde{x}_I(t) \\ \tilde{p}_I(t) \end{pmatrix} + M_{g_a,g_b}^{\Omega_a,\Omega_{II}}(t) \begin{pmatrix} \tilde{x}_{II}(t) \\ \tilde{p}_{II}(t) \end{pmatrix} \right] + \begin{pmatrix} \mathcal{N}_{x,I}(t) \\ \mathcal{N}_{p,I}(t) \end{pmatrix},
$$

$$
\begin{pmatrix} \dot{\tilde{x}}_{II}^{\text{fin}}(t) \\ \dot{\tilde{p}}_{II}^{\text{fin}}(t) \end{pmatrix} = \frac{\kappa}{T} \left[M_{g_b,g_a}^{\Omega_b,\Omega_{II}}(t) \begin{pmatrix} \tilde{x}_{II}(t) \\ \tilde{p}_{II}(t) \end{pmatrix} + M_{g_b,g_a}^{\Omega_b,\Omega_{I}}(t) \begin{pmatrix} \tilde{x}_{I}(t) \\ \tilde{p}_{I}(t) \end{pmatrix} \right] + \begin{pmatrix} \mathcal{N}_{x,II}(t) \\ \mathcal{N}_{p,II}(t) \end{pmatrix},
$$

where the coupling matrix $M_{g_1,g_2}^{\Omega_1,\Omega_2}(t)$ is defined by

$$
M_{g_1,g_2}^{\Omega_1,\Omega_2}(t) = \begin{pmatrix} -g_1 \sin(\Omega_1 t)\sin(\Omega_2 t) & g_1 \sin(\Omega_1 t)\cos(\Omega_2 t) \\ -g_2 \cos(\Omega_1 t)\sin(\Omega_2 t) & g_2 \cos(\Omega_1 t)\cos(\Omega_2 t) \end{pmatrix}.
$$

$\mathcal{N}_{x,I/II}$ and $\mathcal{N}_{p,I/II}$ are noise terms due to the mapping of the input light field onto the atomic systems,

$$
\begin{pmatrix} \mathcal{N}_{x,I/II}(t) \\ \mathcal{N}_{p,I/II}(t) \end{pmatrix} = \frac{1}{\sqrt{T}} \begin{pmatrix} g_{a/b}\sin(\Omega_{a/b} t) & \kappa\cos(\Omega_{I/II} t) \\ g_{b/a}\cos(\Omega_{b/a} t) & \kappa\sin(\Omega_{I/II} t) \end{pmatrix} \begin{pmatrix} \bar{x}_L(ct,0) \\ \bar{p}_L(ct,0) \end{pmatrix}.
$$

We use here continuous light modes with quadratures $\bar{x}_L(ct,0) = x_{L,n}^{\text{in}}/\sqrt{\tau}$ and $\bar{p}_L(ct,0) = p_{L,n}^{\text{in}}/\sqrt{\tau}$ [33]. As shown in the Supplemental Material (SM), the differential equations above can be approximated by their coarse-grained version using coarse-graining time intervals $\Delta t \gg \Omega_{I/II}^{-1}, |\Omega_I - \Omega_{II}|^{-1}$. In this limit, the coupling matrices $M_{g_1,g_2}^{\Omega_1,\Omega_2}$ lead to a negligible contribution for $\Omega_1 \neq \Omega_2$, since their matrix elements average out. Similarly, they can be approximated by a constant diagonal matrix for $\Omega_1 = \Omega_2$ [34]. The noise terms $\mathcal{N}_{x,I/II}(t)$ and $\mathcal{N}_{p,I/II}(t)$ give rise to four modes, which are approximately independent for $\Delta t \gg \Omega_{I/II}^{-1}, |\Omega_I - \Omega_{II}|^{-1}$, and can therefore be squeezed simultaneously such that their contribution becomes negligible. A detailed analysis is provided in [34]. For establishing an effective interaction according to Eq. (5) with $\gamma = \frac{\kappa g}{2T}$, we consider the case $\Omega_{a/b} = \Omega_{II/I}$, $g_{a/b} = -gZ^{\mp 1}$, which leads to

$$
\begin{pmatrix} \dot{x}_I \\ \dot{p}_I \end{pmatrix} = \frac{\kappa g}{2T} \begin{pmatrix} \frac{1}{Z} x_{II} \\ -Z p_{II} \end{pmatrix}, \qquad \begin{pmatrix} \dot{x}_{II} \\ \dot{p}_{II} \end{pmatrix} = \frac{\kappa g}{2T} \begin{pmatrix} Z x_I \\ -\frac{1}{Z} p_I \end{pmatrix}.
$$

By tuning the feedback parameters $g_{a/b}$, it is therefore possible to realize any time evolution, which corresponds to a quadratic interaction Hamiltonian.

As explained above, the setup presented here can not only be applied for facilitating an interaction between remote systems: using a modified configuration, a continuous quantum teleportation can be realized. A teleportation scheme [19] involves three parties, - Alice, Bob and Charlie. It allows Alice to teleport an unknown quantum state provided by Charlie to Bob. In our case, Charlie's state is stored in ensemble II and teleported to ensemble I, which represents Bob, while the light field plays
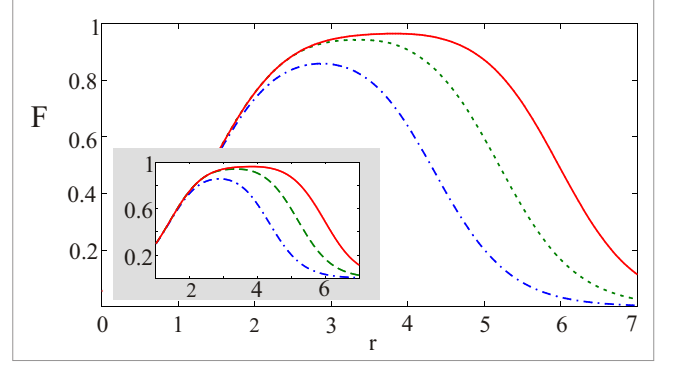

FIG. 2: Fidelity F versus squeezing parameter r of the light field for $R = 3$ and $\Omega_2 = 2\Omega_1$. The three curves correspond in ascending order to $\Omega_1 \Delta t = 100$ (blue dash-dotted line), $\Omega_1 \Delta t = 300$ (dashed green line) and $\Omega_1 \Delta t = 500$ (solid red line). The gain is chosen such that the resulting interaction strength corresponds to the underlying QND interaction ($g = \kappa = 1$). The main panel and the inset show the attainable fidelity for realizing a two mode squeezing and a beam splitter interaction respectively. Any time evolution under a quadratic Hamiltonian can be realized by combining these two interactions with fast local rotations [S2, S3].

the role of Alice. Step (i) in the standard protocol outlined in the introduction corresponds to the interaction between the light field and ensemble I which results in an entangled state. The distribution of entanglement between the two remote sites is realized by the free propagation of the photonic field. Step (ii) corresponds to the interaction of the light with the ensemble II and the measurement of the x-quadrature of the transmitted light. Step (iii) is implemented in the form of a feedback operation realizing a conditional displacement on ensemble I, which can be done using magnetic fields. Using this protocol, the deterministic teleportation of a quantum state between two atomic ensembles has been demonstrated recently [35]. We present now an extension to a time-continuous operation including real-time feedback, which facilitates the teleportation of quantum dynamics. For this purpose, we consider the special case of the scheme introduced above where $\Omega_I = \Omega_{II} = \Omega$ and Feedback is applied only to the first ensemble $g_{x,I}(t) = \frac{1}{\sqrt{N}} \bar{g}_x \sin(\Omega t)$, $g_{p,I}(t) = \frac{1}{\sqrt{N}} \bar{g}_p \cos(\Omega t)$. With this choice, Eq. (4) yields

$$
\begin{pmatrix} \dot{\tilde{x}}_I^{\text{fin}}(t) \\ \dot{\tilde{p}}_I^{\text{fin}}(t) \end{pmatrix} = \frac{\kappa}{T} M_{\bar{g}_x,\bar{g}_p}^{\Omega,\Omega}(t) \begin{pmatrix} \tilde{x}_I(t) \\ \tilde{p}_I(t) \end{pmatrix} + \frac{\kappa}{T} M_{\bar{g}_x,\bar{g}_p}^{\Omega,\Omega}(t) \begin{pmatrix} \tilde{x}_{II}(t) \\ \tilde{p}_{II}(t) \end{pmatrix}
$$
$$
+ \frac{1}{\sqrt{T}} \begin{pmatrix} \bar{g}_x \sin(\Omega t) & \kappa \cos(\Omega t) \\ \bar{g}_p \cos(\Omega t) & \kappa \sin(\Omega t) \end{pmatrix} \begin{pmatrix} \bar{x}_L(ct,0) \\ \bar{p}_L(ct,0) \end{pmatrix}. \quad (6)
$$

The second term in Eq. (6) allows us to control Bob's dynamics using Charlie's system. In order to demonstrate the transmission of a time evolution, we add an extra Hamiltonian corresponding to a time dependent transverse magnetic field on Charlie's side $H_{\text{extra,II}} = \alpha_x(t)x_{II} + \alpha_p(t)p_{II}$. $\alpha_x(t)$, where $\alpha_p(t)$ are real time de-

pendent functions. Accordingly,

$$\begin{pmatrix} \tilde{x}_{\mathrm{II}}(t) \\ \tilde{p}_{\mathrm{II}}(t) \end{pmatrix} = \int_0^t dt' \begin{pmatrix} \tilde{\alpha}_p(t') \\ -\tilde{\alpha}_x(t') \end{pmatrix} + \begin{pmatrix} \tilde{x}_{\mathrm{II}}^{\mathrm{in}} \\ \tilde{p}_{\mathrm{II}}^{\mathrm{in}} \end{pmatrix} \qquad (7)$$
$$+ \frac{\kappa}{\sqrt{T}} \int_0^t dt' \begin{pmatrix} \cos(\Omega t') \\ \sin(\Omega t') \end{pmatrix} \bar{p}_{\mathrm{L}}(ct', 0),$$

where $\tilde{\alpha}_x(t)$ and $\tilde{\alpha}_p(t)$ describe the time dependence of $H_{\mathrm{extra,II}}$ in the rotating frame (see Eq. (1)). The resulting time evolution on Bob's side can be evaluated by inserting Eq. (7) into Eq. (6). As in the general case discussed above, the resulting differential equation can be approximated by the coarse-grained equation with $M_{g,-g}^{\Omega,\Omega} \to -\frac{1}{2}\mathbb{1}$ for $\Omega\Delta t \gg 1$, where Δt is the coarse graining time interval. This way, one obtains

$$\begin{pmatrix} \dot{\tilde{x}}_{\mathrm{I}}^{\mathrm{fin}}(t) \\ \dot{\tilde{p}}_{\mathrm{I}}^{\mathrm{fin}}(t) \end{pmatrix} = \frac{\kappa}{2T} \int_0^t dt' \begin{pmatrix} -\bar{g}_{\mathrm{x}} \tilde{\alpha}_p(t) \\ \bar{g}_{\mathrm{p}} \tilde{\alpha}_x(t) \end{pmatrix} + \frac{\kappa}{2T} \begin{pmatrix} -\bar{g}_{\mathrm{x}}[\tilde{x}_{\mathrm{I}}^{\mathrm{fin}}(t) + \tilde{x}_{\mathrm{II}}^{\mathrm{in}}(t)] \\ \bar{g}_{\mathrm{p}}[\tilde{p}_{\mathrm{I}}^{\mathrm{fin}}(t) + \tilde{p}_{\mathrm{II}}^{\mathrm{in}}(t)] \end{pmatrix}$$
$$+ \frac{\kappa^2}{2T^{\frac{3}{2}}} \int_0^t dt' \begin{pmatrix} -\bar{g}_{\mathrm{x}} \cos(\Omega t) \\ \bar{g}_{\mathrm{p}} \sin(\Omega t) \end{pmatrix} \bar{p}_{\mathrm{L}}(ct, 0)$$
$$+ \frac{1}{\sqrt{T}} \begin{pmatrix} -\bar{g}_{\mathrm{x}} \sin(\Omega t) & \kappa \cos(\Omega t) \\ \bar{g}_{\mathrm{p}} \cos(\Omega t) & \kappa \sin(\Omega t) \end{pmatrix} \begin{pmatrix} \bar{x}_{\mathrm{L}}(ct, 0) \\ \bar{p}_{\mathrm{L}}(ct, 0) \end{pmatrix}. \qquad (8)$$

The first term on the right side is equivalent to an effective field experienced by Bob. More specifically, Bob's ensemble evolves as if it was placed in a transverse magnetic field, whose time dependence is given by Charlie's evolution (see Eq. (7)). This way, the effect of the magnetic field applied at Charlie's side is teleported to Bob. This is possible, since entanglement generation, measurement and feedback are performed continually. In contrast to a traditional teleportation, Charlie's quantum state is therefore not destroyed in a single step and subsequently restored on Bob's side, but rather transmitted continuously, which offers the possibility to include the effect of a time evolution.

In the remainder of the article, we discuss imperfections. To this end, we consider the case, where real-time feedback is applied to both ensembles. The following discussion can also be applied to evaluating the added noise in the teleportation protocol. For analyzing the performance of the scheme, we introduce a figure of merit which quantifies the deviation of the realized time evolution from the desired one using the Jamiolkowsi isomorphism between quantum maps and states. Both, the imperfect real map and the ideal one are transformed into their corresponding states and then compared as explained in the SM. More specifically, the time evolution ε_T acting on the atomic system is described in terms of an entangled state of twice the system size that can be used to teleport a given input state ρ_{in} through ε_T, such that the output $\varepsilon_T(\rho_{\mathrm{in}})$ is obtained. This entangled state consists of two copies of a two mode squeezed state with squeezing parameter R. For $R \to \infty$, which corresponds to a quantum state with infinite energy, any input state can be teleportet through ε_T. For finite R, this holds for a restricted set of input states. We start by considering the
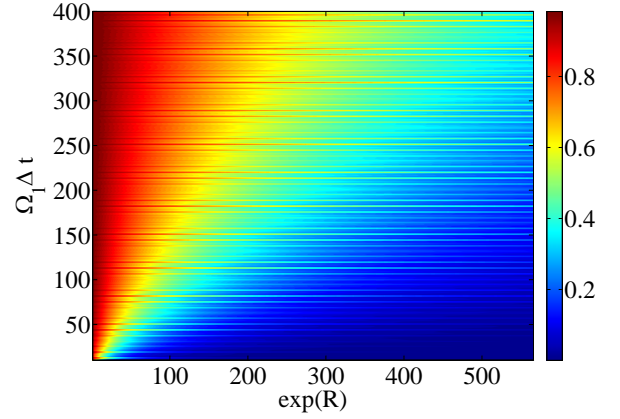

FIG. 3: Fidelity for optimal squeezing r_{opt} versus e^R and $\Omega_1\Delta t$ for $\Omega_2 = 2\Omega_1$. As in Fig. 2, $g = \kappa = 1$. This plot shows the fidelity for realizing a two mode squeezing interaction. The results for a beamsplitter interaction are provided in [34].

fidelity for fixed R. As explained in the SM, the scheme involves two types of imperfections, fast rotating terms in the evolution and light noise added to the atomic system. If a time window $\Delta t \gg \Omega_1, \Omega_2, |\Omega_1 - \Omega_2|$ is considered, the former are negligible and the latter can be suppressed using squeezed input light fields. Fig. 2 shows the attainable fidelity for fixed R, Ω_1 and $\Omega_2 = 2\Omega_1$ versus the squeezing of the light field r for different values of $\Omega_1\Delta t$. For increasing $\Omega_1\Delta t$, the optimal squeezing parameter r_{opt} increases, which leads to an increased accuracy of the scheme. For $\Omega_1\Delta t \to \infty$, $r_{\mathrm{opt}} \to \infty$ and $F \to 1$. For increasing values of the parameter R, correspondingly high values of Δt are required to obtain a good fidelity. The required value of Δt increases with e^R as shown in Fig. 3 which displays the fidelity F for fixed Larmor frequencies Ω_1 and $\Omega_2 = 2\Omega_1$. This graph features a fine substructure since for time windows $\Delta t = 2\pi/\Omega_1$, local maxima are obtained, which gives rise to stripe-like regions of high fidelity. In general, the realization of a desired time evolution with a good temporal resolution requires high Larmor frequencies. For fixed Ω_1 and Ω_2, the attainable precision depends on the temporal resolution as shown in Fig. 2 and Fig. 3. We remark that higher fidelities can be obtained if a stroboscopic interaction is implemented where points of interest in time are chosen to coincide with the local maxima.

In conclusion, we have shown how a time evolution can be teleported by means of a dynamical teleportation scheme, where entangling operations, measurement and feedback are performed continuously and simultaneously. Moreover, we demonstrated how a generalized version of this protocol can be used to implement arbitrary quadratic interactions between two separate systems. In particular, important tools for QIS, such as a purely active (entangling) or a purely passive (quantum state transfer) interaction can be obtained.

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[25] We neglect the distance R between the ensembles since it is assumed that the time delay $\delta t = R/c$ is short on the time scale $1/\Gamma_{\text{eff}}$ on which atomic transitions take place, such that retardation effects can be ignored. For the setup under consideration [2], where the distance R between the ensembles is on the order of a meter and $\Gamma_{\text{eff}} \ll \Gamma$ (where Γ is the natural linewidth of the involved excited states), this condition is very well fullfilled. In our protocols, non-negligible retardation effects would lead to differential equations which are non-local in time.

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[29] We assume here that the rotation of the atomic system due to the magnetic field is negligible during the short time interval τ (compare [40]).

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[31] We consider here only interaction terms. Local terms can be either or passive or of active type. The former can be realized by applying magnetic fields, while the latter can for example be accomplished by combining a QND light-matter interaction with continuous measurements on the light field and real-time feedback.

[32] The two mode squeezing Hamiltoninan H_A is given by $H_A \propto i \left(a_1 a_2 - a_1^\dagger a_2^\dagger \right)$ where $a_j = \frac{1}{\sqrt{2}} \left(x_j + i p_j \right)$, with $j = 1, 2$. $H_P \propto i \left(a_1 a_2^\dagger - a_1^\dagger a_2 \right)$ describes a beamsplitter-like interaction.

[33] The continuous light modes with quadratures $\bar{x}_L(\xi, t)$, $\bar{p}_L(\xi, t)$ are obtained by considering spatially localized modes $x_L(z, t)$, $p_L(z, t)$ with $[x_L(z, t), x_L(z', t)] = ic\delta(z - z')$ and performing the variable transformation $\xi = ct - z$ (such that $\bar{x}_L(\xi, t) = x_L(ct - \xi, t)$).

[34] Supplemental Material

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Supplemental Material

In the following, we discuss imperfections of the proposed scheme for inducing non-local dynamics and show that the desired time evolution can be realized perfectly in the ideal case. In Sec. 1, we consider the different types of imperfection and explain in which parameter regime the desired time evolution can be faithfully implemented. In Sec. 2 we show how the deviation of the implemented time evolution from the desired one can be quantified and introduce an adequate figure of merit. In Sec. 3, we include squeezed light in the consideration and calculate the resulting fidelity of the atomic evolution numerically. We consider here the general scheme involving feedback on both ensembles for generating an effective interaction between two atomic ensembles. The noise contribution in the teleportation scheme can be analyzed along the same lines [S1] as the added noise discussed in the following (in particular in Sec. 3).

1. Imperfections of the scheme

Throughout the Supplemental Material, we consider two different time scales

$$\tau \ll \Omega^{-1} \ll \Delta t,$$

and assume that the frequencies Ω_1, Ω_2, and $|\Omega_1 - \Omega_2|$ take values of the same order of magnitude, which defines a time scale Ω^{-1}. $\tau \ll \Omega^{-1}$ is an infinitesimally short time interval which will be used to express a continuous time evolution in a discretized form. $\Delta t \gg \Omega^{-1}$ is length of the considered time window.

We consider the setup shown in Fig. 1a in the main text for establishing an effective interaction between two atomic ensembles. The time evolution of the atomic system during an infinitesimal time step of duration τ is given by

$$\begin{pmatrix} \tilde{x}_{\mathrm{I}}([n+1]\tau) \\ \tilde{p}_{\mathrm{I}}([n+1]\tau) \\ \tilde{x}_{\mathrm{II}}([n+1]\tau) \\ \tilde{p}_{\mathrm{II}}([n+1]\tau) \end{pmatrix} = (\mathbb{1} + G(n\tau)\,\tau) \begin{pmatrix} \tilde{x}_{\mathrm{I}}(n\tau) \\ \tilde{p}_{\mathrm{I}}(n\tau) \\ \tilde{x}_{\mathrm{II}}(n\tau) \\ \tilde{p}_{\mathrm{II}}(n\tau) \end{pmatrix} + \begin{pmatrix} N_{x_1,n} \\ N_{p_1,n} \\ N_{x_2,n} \\ N_{p_2,n} \end{pmatrix}. \tag{S.1}$$

The second term on the right is an undesired contribution due to the mapping of the input-quadratures of the light field onto the atomic ensembles,

$$\begin{pmatrix} N_{x_1,n} \\ N_{p_1,n} \\ N_{x_2,n} \\ N_{p_2,n} \end{pmatrix} = \frac{\kappa}{\sqrt{N}} \begin{pmatrix} \cos(\Omega_1 t) \\ \sin(\Omega_1 t) \\ \cos(\Omega_2 t) \\ \sin(\Omega_2 t) \end{pmatrix} p_{L,n}^{\mathrm{in}} - \frac{g}{\sqrt{N}} \begin{pmatrix} \frac{1}{Z}\sin(\Omega_2 t) \\ Z\cos(\Omega_2 t) \\ Z\sin(\Omega_1 t) \\ \frac{1}{Z}\cos(\Omega_1 t) \end{pmatrix} x_{L,n}^{\mathrm{in}}. \tag{S.2}$$

The matrix G is given by

$$G(t) = \frac{g\kappa}{T} \begin{pmatrix} \frac{1}{Z}\sin(\Omega_2 t)\sin(\Omega_1 t) & -\frac{1}{Z}\sin(\Omega_2 t)\cos(\Omega_1 t) & \frac{1}{Z}\sin(\Omega_2 t)\sin(\Omega_2 t) & -\frac{1}{Z}\sin(\Omega_2 t)\cos(\Omega_2 t) \\ Z\cos(\Omega_2 t)\sin(\Omega_1 t) & -Z\cos(\Omega_2 t)\cos(\Omega_1 t) & Z\cos(\Omega_2 t)\sin(\Omega_2 t) & -Z\cos(\Omega_2 t)\cos(\Omega_2 t) \\ Z\sin(\Omega_1 t)\sin(\Omega_1 t) & -Z\sin(\Omega_1 t)\cos(\Omega_1 t) & Z\sin(\Omega_1 t)\sin(\Omega_2 t) & -Z\sin(\Omega_1 t)\cos(\Omega_2 t) \\ \frac{1}{Z}\cos(\Omega_1 t)\sin(\Omega_1 t) & -\frac{1}{Z}\cos(\Omega_2 t)\cos(\Omega_1 t) & \frac{1}{Z}\cos(\Omega_2 t)\sin(\Omega_2 t) & -\frac{1}{Z}\cos(\Omega_2 t)\cos(\Omega_2 t) \end{pmatrix}.$$

The dynamics described by Eq. (S.1) has to be compared to the desired time evolution for the atomic system,

$$\dot{\vec{X}}(t) = G_0 \vec{X}(t)$$

where $\vec{X} = (\tilde{x}_{\mathrm{I}}, \tilde{p}_{\mathrm{I}}, \tilde{x}_{\mathrm{II}}, \tilde{p}_{\mathrm{II}})^T$ and

$$G_0 = \frac{g\kappa}{2T} \begin{pmatrix} 0 & 0 & \frac{1}{Z} & 0 \\ 0 & 0 & 0 & -Z \\ Z & 0 & 0 & 0 \\ 0 & -\frac{1}{Z} & 0 & 0 \end{pmatrix}.$$

The scheme involves two types of imperfections. Firstly, the proposed scheme does not implement exactly the atomic time evolution matrix G_0 but rather the time dependent matrix $G(t)$. Secondly, light noise is added to the system (see

Eq. (S.2)). We start by discussing the first type of imperfection. To this end, we are considering the time evolution during a time window Δt in the absence of added noise. The solution of the differential equation $\dot{\vec{X}}(t) = G(t)\vec{X}(t)$ can be approximated by $\vec{X}(t) = e^{G_0 t}\vec{X}(0)$ in the limit

$$\begin{aligned}
\epsilon_{1,1} &= (\Omega_1 \Delta t)^{-1} \ll 1, \\
\epsilon_{2,2} &= (\Omega_2 \Delta t)^{-1} \ll 1, \\
\epsilon_{1,2} &= (|\Omega_1 - \Omega_2|\Delta t)^{-1} \ll 1.
\end{aligned}$$

This is due to the fact that the matrix elements involving two different trigonometric functions average out upon integration, for example

$$\frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \sin(\Omega_i t')\cos(\Omega_i t') = \mathcal{O}(\epsilon_{i,i}), \qquad \frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \sin(\Omega_i t')\sin(\Omega_i t') = \mathcal{O}(\epsilon_{i,j}), \text{ for } i \neq j,$$

while

$$\frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \sin^2(\Omega_i t') = \frac{1}{2} + \mathcal{O}(\epsilon_{i,i}), \qquad \frac{1}{\Delta t}\int_t^{t+\Delta t} dt' \cos^2(\Omega_i t') = \frac{1}{2} + \mathcal{O}(\epsilon_{i,i}).$$

The effective time evolution is valid on a coarse-grained time scale $\Delta t \gg \Omega_1^{-1}, \Omega_1^{-2}, |\Omega_1 - \Omega_2|^{-1}$. This can be understood in analogy to familiar examples of effective 2$^{\text{nd}}$ order Hamiltonians such as AC stark shift Hamiltonians.

As a next step, we include the noise terms in the discussion and consider the complete time evolution $\dot{\vec{X}}(t) = G(t)\vec{X}(t) + \vec{N}(t)$ and its solution

$$\vec{X}(t) = \mathcal{T}e^{\int_0^{\Delta t} G(\tau)d\tau}\,\vec{X}(0) + \vec{N}^{\text{int}}, \tag{S.3}$$

where $\mathcal{T}$ is the time ordering operator and

$$\vec{N}^{\text{int}} = \int_0^{\Delta t} \mathcal{T}e^{\int_\tau^{\Delta t} G(\tau')d\tau'}\,\vec{N}(\tau')d\tau'.$$

The components of the integrated noise operator $\vec{N}^{\text{int}} = \begin{pmatrix} N_{x1}^{\text{int}} & N_{p1}^{\text{int}} & N_{x2}^{\text{int}} & N_{p2}^{\text{int}} \end{pmatrix}^T$ are mutually independent

$$[N_i^{\text{int}}, N_j^{\text{int}}] = 0,$$

for $\epsilon \to 0$, where $\epsilon = \max(\epsilon_{1,1}, \epsilon_{2,2}, \epsilon_{1,2})$. Therefore, all four noise contributions can be squeezed simultaneously in this limit. Note that the two types of imperfections that occur in this scheme are intimately linked. For $\epsilon \to 0$, the first part of the differential equation corresponds to a unitary time evolution, which implies that the noise terms can be squeezed such that their contribution to the dynamics becomes negligible.

2. Comparison of time evolutions

In order to asses the performance of the scheme, we need to quantify how close the established time evolution is to the desired one. We do this in terms of covariance matrices using the Jamiolkowski isomorphism as explained in Sec. 2 a. The corresponding figure of merit is introduced in Sec. 2 b.

a. *Characterization of linear time evolutions using The Jamiolkowski isomorphism*

The Jamiolkowski isomorphism allows one to describe a time evolution in terms of states. Gaussian quantum states, i.e. states with a Gaussian Wigner function, are completely characterized by their first and second moments. They can be conveniently described in terms of their displacement vector D, with components $D_i = \langle X_i \rangle$, and their covariance matrix Γ, with $\Gamma_{ij} = \langle \{X_i, X_j\}_+ \rangle$, where $\{\cdot, \cdot\}_+$ is the anticommutator [S4]. In the following, we use the Jamiolkowski isomorphism in order to characterize a time evolution of the atomic system in terms of a quantum state (Γ, D), as
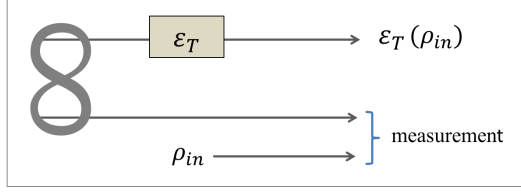
Supplementary figure S.1: Characterization of a completely positive map ϵ_T in terms of a quantum state using the Jamiolkowski isomorphism. An operation ϵ_T acting on quantum system can be represented by the density matrix ρ_J, which corresponds to twice the system size. ρ_J can be interpreted as two entangled copies of a quantum state, where ϵ_T has been applied to one subsystem. Given the state ρ_J, the map ϵ_T can be (probabilistically) implemented by teleporting an input state ρ_{in} through ϵ_T. This can be done by performing a suitable joined measurement on ρ_{in} and one subsystem.

explained in [S5]. To this end, we consider two identical two mode squeezed states with squeezing parameter R. The corresponding covariance matrix Γ_J for the two entangled copies of the atomic system is given by

$$\Gamma_J = \begin{pmatrix} A(R) & C(R) \\ C(R)^T & A(R) \end{pmatrix},$$

where $A(R)$ and $C(R)$ are 4×4 matrices,

$$A(R) = \cosh(R)\mathbb{1}, \qquad C(R) = \sinh(R)\mathrm{diag}(1, -1, 1, -1).$$

In the limit $R \to \infty$, the two mode squeezed state is the improper maximally entangled state $|\Psi_{\mathrm{TMSS},\infty}\rangle \propto \sum_{n=0}^{\infty} |n, n\rangle$. A time evolution of the atomic system $\vec{X}(t) = S(t)\vec{X}(0) + n(t)$ (where $S(t)$ is a time evolution matrix and $n(t)$ added noise) can be characterized by applying the corresponding completely positive map ε_T to one part of the entangled composite quantum state introduced above, which results in the covariance matrix

$$\Gamma_{J[S,n]} = \begin{pmatrix} SA(R)S^T + \Gamma_{\mathrm{noise}} & SC(R) \\ (DC(R))^T & A(R) \end{pmatrix},$$

where $\Gamma_{\mathrm{noise}} = nn^T + n^T n$. Fig. S.1 illustrates an intuitive explanation of this characterization. The time evolution ε_T can be applied to an input state ρ_{in} by teleporting this state by means of a joint measurement with one part of the composite entangled system (see [S5] for a more detailed explanation). For $R \to \infty$, any input state ρ_{in} can be probabilistically teleportet in this way, resulting in the output $\varepsilon_T(\rho_{\mathrm{in}})$. For finite values of R, this reasoning can be applied to a restricted set of input states.

A noisefree time evolution ε_T translates into a Jamiolkowski covariance matrix Γ_J that corresponds to a pure state. Deviations of the implemented time evolution from the desired unitary one translate directly into noise added to the Jamiolkowski matrix.

b.   Figure of merit

In order to assess the deviation of the covariance matrix that corresponds to the time evolution implemented by the proposed scheme $\Gamma_{\Delta t}$, from the ideal (desired) one, Γ_{∞}, we introduce the figure of merit

$$E = \max_{\vec{C}} \left[\frac{\vec{C}^{\dagger}(\Gamma_{\Delta t} - \Gamma_{\infty})\vec{C}}{\vec{C}^{\dagger}\Gamma_{\Delta t}\vec{C}} \right],$$

which quantifies the error. Maximization over all complex vectors $\vec{C}$ yields

$$E = 1 - m(\Gamma_E), \tag{S.4}$$

where $m(\Gamma_E)$ is the minimum eigenvalue of the matrix

$$\Gamma_E = \frac{1}{\sqrt{\Gamma_{\Delta t}}} \Gamma_{\infty} \frac{1}{\sqrt{\Gamma_{\Delta t}}}. \tag{S.5}$$

This can be seen by noting that

$$E = 1 - \min_{\vec{C}} \left[\frac{\vec{C}^\dagger \Gamma_\infty \vec{C}}{\vec{C}^\dagger \Gamma_{\Delta t} \vec{C}} \right],$$

and considering the derivative of the expression $\lambda_\infty / \lambda_{\Delta t}$ with respect to $\vec{C}$, where

$$\lambda_{\Delta t} = \vec{C}^\dagger \Gamma_{\Delta t} \vec{C}, \qquad \lambda_\infty = \vec{C}^\dagger \Gamma_\infty \vec{C}.$$

The resulting condition for extremality

$$\lambda_\infty \Gamma_{\Delta t} \vec{C} - \lambda_{\Delta t} \Gamma_\infty \vec{C} = 0$$

translates directly into

$$\frac{\lambda_{\Delta t}}{\lambda_\infty} \vec{d} = \Gamma_E \vec{d},$$

where Γ_E is given by Eq. (S.5) and $\vec{d} = \sqrt{\Gamma_{\Delta t}} \vec{C}$.

3. Fidelity of the scheme for squeezed input states

In this section, we show that the desired time evolution can be implemented perfectly if squeezed light fields are used. To this end, we explain how squeezed photonic input states can be included in the derivation and calculate the attainable fidelities.

The time evolution of the atomic system is given by Eq. (S.3). The total (integrated) noise contribution $N^{\mathrm{int}} = (N_{x1}^{\mathrm{int}}, N_{p1}^{\mathrm{int}}, N_{x2}^{\mathrm{int}}, N_{p2}^{\mathrm{int}})^T$ can be written in the form

$$\vec{N}^{\mathrm{int}} = T \vec{Y},$$

where $\vec{Y} = (x_{\mathrm{L},1}, x_{\mathrm{L},2}, ..., x_{\mathrm{L},N}, p_{\mathrm{L},1}, p_{\mathrm{L},2}, ..., p_{\mathrm{L},N})^T$ is the input light field vector and T is a $4 \times 2N$ matrix. We can define canonical partners $\vec{N'}^{\mathrm{int}} = (N'^{\mathrm{int}}_{x1}, N'^{\mathrm{int}}_{p1}, N'^{\mathrm{int}}_{x2}, N'^{\mathrm{int}}_{p2})^T$ and normalize the operators such that

$$[N_i^{\mathrm{int}}, N'^{\mathrm{int}}_j] = i\, \delta_{i,j}, \qquad i, j = 1, 2, 3, 4.$$

Using these definitions, we can define the vector $\vec{N}_8^{\mathrm{int}} = \left(N_{x1}^{\mathrm{int}}, N_{p1}^{\mathrm{int}}, N_{x2}^{\mathrm{int}}, N_{p2}^{\mathrm{int}}, N'^{\mathrm{int}}_{x1}, N'^{\mathrm{int}}_{p1}, N'^{\mathrm{int}}_{x2}, N'^{\mathrm{int}}_{p2} \right)^T$, where the subscript indicates that this vector includes the canonical partners. The vector $\vec{N}_8^{\mathrm{int}}$ can be written in matrix form

$$\vec{N}_8^{\mathrm{int}} = T_8 \vec{Y}.$$

The noise operators contribution to the atomic evolution N_{x1}^{int}, N_{p1}^{int}, N_{x2}^{int}, N_{p2}^{int} are independent for $\epsilon \to 0$. In this limit, the total noise commutation relation

$$[N_{8,i}^{\mathrm{int}}, N_{8,j}^{\mathrm{int}}] = i\, \delta_{i,j}$$

is given by

$$\sigma_\infty = \begin{pmatrix} 0_4 & \mathbb{1}_4 \\ -\mathbb{1}_4 & 0_4 \end{pmatrix},$$

where 0_4 and $\mathbb{1}_4$ are the 4×4 zero and unit matrix respectively. For finite values of ϵ,

$$\sigma_{\Delta t} = T_8\, \sigma_\infty T_8^T = \sigma_\infty + O(\epsilon).$$

Due to the small correction on the right side, the noise modes can not be squeezed simultaneously. We consider here a squeezing operation with squeezing parameter r on the input light field which is applied prior to the actual protocol. More specifically, we consider the operation on the input light field that corresponds to a squeezing of the
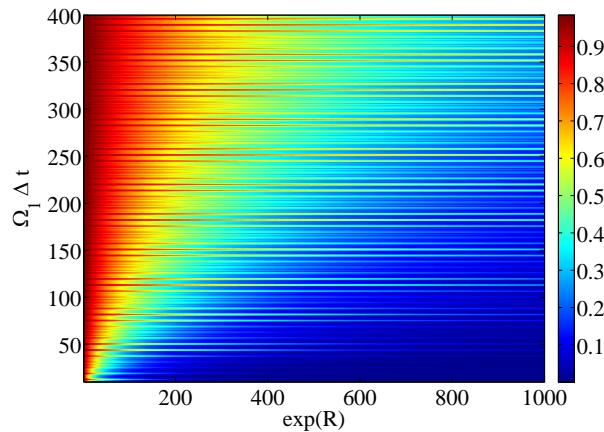
Supplementary figure S.2: Fidelity for optimal squeezing parameter r_{opt} of the input light field versus e^R and $\Omega_1\Delta t$ for fixed Larmor frequencies Ω_1 and $\Omega_2 = 2\Omega_1$. As in Fig. 2 in the main text, the gain is chosen such that the resulting interaction strength corresponds to the underlying QND interaction ($g = \kappa = 1$). This plots shows the attainable fidelity for realizing a beamsplitter interaction.

integrated noise modes [S6]. The fact that $\sigma_{\Delta t} \neq \sigma_\infty$ leads to a tradeoff that prevents the fidelity of the scheme to grow continuously with increasing squeezing parameter r. Therefore, there is an optimal squeezing parameter r_{opt} for a given value of ϵ. This can be understood by expressing the integrated noise modes in terms of canonically commuting modes $\vec{n}_8 = (n_1, n_2, n_3, n_4, n_1', n_2', n_3', n_4')^T$ (with $[n_i, n_j] = i(\sigma_\infty)_{ij}$), such that the noise contribution to the atomic covariance matrix can be written as

$$\Gamma_{\mathrm{noise}} = \alpha\,\Gamma_n + \beta\,\Gamma_{n'} \ ,$$

with real coefficients α and β, where β is on the order of ϵ. Γ_n and $\Gamma_{n'}$ are the covariance matrices corresponding to $\vec{n} = (n_1, n_2, n_3, n_4)^T$ and $\vec{n}' = (n_1', n_2', n_3', n_4')^T$ respectively. Considering a squeezed state of the modes $\vec{n}$ with $\Gamma_n = e^{-r}\mathbb{1}_4$ and $\Gamma_{n'} = e^r\mathbb{1}_4$ leads to the tradeoff described above. Fig. 2 in the main text shows the fidelity $F = 1 - E$ (where E is the error as defined in Sec. 2 b) versus the input squeezing r for fixed values of Ω_1, Ω_2 and R. We analyze here the realization of an active (two mode squeezing) and a passive (beamsplitter like) interaction, since any evolution under a quadratic Hamiltonian can be realized by combining these two interactions with local rotations [S2, S3]. The plot shows how decreasing values of ϵ lead to an increased optimal squeezing parameter r_{opt} and therefore to an increased accuracy of the scheme.

For high values of the parameter R, correspondingly high values of Δt are required to obtain a good fidelity according to the definition given in Sec. 2 b. More specifically, the optimal fidelity depends on the product $e^R\Delta T^{-1}$ as shown in Fig. S.2 which depicts $F = 1 - E$ versus e^R and Δt for fixed Larmor frequencies Ω_1 and $\Omega_2 = 2\Omega_1$. Apart from that, the graph features a fine substructure of local maxima for $\Delta t = 2\pi/\Omega_1$.

In summary, high fidelities can be obtained for $\epsilon \ll 1$, which requires $\Delta t \gg \Omega_1, \Omega_2, |\Omega_1 - \Omega_2|$. As outlined in the main text, this relation can be understood in terms of a time coarse graining with a coarse graining time interval Δt. The realization of a desired time evolution with a very good temporal resolution requires high Larmor frequencies. For fixed Larmor frequencies Ω_1 and Ω_2, the attainable precision depends on the temporal resolution as shown in the figures. We remark that higher fidelities can be obtained if one realizes a stroboscopic interaction where points of interest in time are chosen to coincide with the phase matching condition.

[S1] In the teleportation scheme, light noise is added to the atomic system (which is described by the terms in the second and third line in Eq. (8) in the main text). The corresponding integrated noise operators commute in the limit $\Omega\Delta t$ (where Δt is the considered interaction time) if the gain factors gain factors $g_x = -g_p$ are chosen. In this case, the desired time evolution is teleported perfectly with a phase shift, which is not relevant. The discussion in Sec. 3 can be directly applied to the added noise in the teleporation scheme.

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[S4] C. Weedbrook, S. Pirandola, R. Garcia-Patron, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, Rev. Mod. Phys. **84**, 621 (2012).

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[S6] The operation on the input light field that corresponds to a squeezing operation on the integrated noise mode can be described in terms of the Hamiltonian $H = (\vec{N}_8^{\text{int}})^T M_8 \vec{N}_8^{\text{int}}$, where $M_8 = -\frac{r}{2\,T_{\text{squ}}} \begin{pmatrix} 0_4 & \mathbb{1}_4 \\ \mathbb{1}_4 & 0_4 \end{pmatrix}$. T_{squ} is the time period during which the squeezing Hamiltonian is applied. This leads to the differential equation $\dot{\vec{N}}_8^{\text{int}} = \mathcal{G}_8 \vec{N}_8^{\text{int}}$ with $\mathcal{G}_8 = \sigma_{\Delta t}(M_8 + M_8^T)$.